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Cite as: AIP Conference Proceedings **2333**, 090018 (2021); https://doi.org/10.1063/5.0041699
Published Online: 08 March 2021

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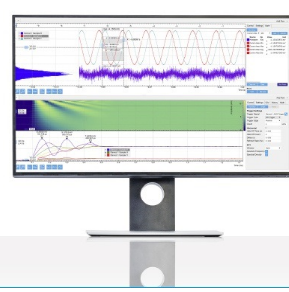

Features of Optimal Control Problem Moving Bodies with Variable Geometry in a Viscous Medium with Non-constant Density

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Abstract. The problems of two types of dynamic optimization of flow around solids are described in general form. The features of these problems are analyzed from the point of view of the optimal control theory, the difficulties caused by such features, and ways to overcome them. The problem solving schemes developed for the applied engineer are briefly described, the idea of the mathematical justification of one of them is presented.

FEATURES STATEMENTS OF DYNAMIC OPTIMIZATION PROBLEMS

Rigid bodies and mechanical systems composed by connecting such bodies, called links, are considered [1], [2]. The phase state of each link can be uniquely determined by a set of generalized coordinates corresponding to the degrees of freedom of the link, and derivatives of the generalized coordinates with respect to time. It is assumed that the phase state of the link can be controlled by forces and moments.

The main content is the study of two types of dynamic flow optimization problems [3]. As will be established, we are talking about a new class of problems that are relevant from the point of view of the theory of singular or special solutions [4] to dynamic optimization problems.

In problems of the first type, it is required to find the laws of change in the control forces and moments that ensure the system moves in a given time from the initial phase state to a given target set with minimal costs for overcoming the resistance forces of the medium. Such problems have the following features. Firstly, they are irregular, unless control actions are explicitly included in the current expression for the power of resistance forces. Indeed, the control forces and moments acting on a mechanical system enter linearly into the equations of its motion. Hence, the Hamiltonian depends on the control forces and moments also linearly. Therefore, the Euler–Lagrange equations do not explicitly contain control actions and, therefore, do not formally determine their optimal values in terms of phase and conjugate variables. Secondly, as experience shows, this is a sure sign that (and so it turned out) that optimal programs for changing control forces and moments have impulse components. Therefore, classical variational tools are not directly applicable for finding optimal programs other than generalizing the Pontryagin maximum principle to the simplest classes of impulse controls [4].

In problems of the second type, it is required to find the laws of change of control forces and moments that ensure the movement of the system for a given time from the initial phase state to a given target set with a minimum value of the work of control forces and moments. In addition to the above two features, such problems have a third, which consists in the difficulty of calculating energy consumption. The fact is that for this it is necessary to determine the correct method of multiplying the impulse control forces and moments by discontinuous implementations of the linear and angular velocities of the system links, respectively.

CONSTRAINTS ON CONTROL FORCES AND MOMENTS

Earlier, the problem of the optimal energy consumption for overcoming the resistance of a viscous medium to move the object from one phase state to another was considered [5], [6]. The problem was investigated in two versions. In the first of them, the Stokes formula was used to calculate the resistance, and the second used the Boussinesq

Applications of Mathematics in Engineering and Economics (AMEE'20)
AIP Conf. Proc. 2333, 090018-1–090018-4; https://doi.org/10.1063/5.0041699
Published by AIP Publishing. 978-0-7354-4077-7/$30.00

formula [7], [8] which takes into account the unsteady flow effects. It turned out that the hypothesis of quasistationary flow around leads to a relative error in the optimal energy consumption of only about 0.02 % [9].

This result was the basis for the study of all other problems in the framework of the following restrictions on permissible control forces and moments.

Constraint 1. Fluid is incompressible.

With account of the equation of continuity, this constraint is equivalent to zero velocity of the volume strain

$$\mathrm{div}\mathbf{v} = 0. \tag{1}$$

Constraint 2. The generalized Newton hypothesis is fulfilled

$$P = -pE + \mu\Big(\frac{\partial \mathbf{v}}{\partial x} + \Big(\frac{\partial \mathbf{v}}{\partial x}\Big)^*\Big), \tag{2}$$

where P is the linear operator defined by the stress tensor, $p = p(t,x)$ denotes the scalar field of pressure, μ is the dynamic viscosity coefficient, E is the identity mapping, $\dfrac{\partial \mathbf{v}}{\partial x}$ is the Frechet derivative, and $\Big(\dfrac{\partial \mathbf{v}}{\partial x}\Big)^*$ is the conjugate operator.

Let a body of bounded size with sufficiently smooth boundary S move in fluid. One of the fluid mechanics axioms is the sticking condition: at the body surface points the velocity vector of fluid particle is equal to the velocity vector of the corresponding body point. This condition implies that in the case of translational motion of the body the following equality is fulfilled at its surface [10]

$$\Big(\frac{\partial \mathbf{v}}{\partial x}\Big)^* \mathbf{n} = 0, \tag{3}$$

where $\mathbf{n}$ is the unit vector of the outward normal to the surface S at the point x.

Then the Navier–Stokes equation is

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} = -\frac{\partial \hat{\mathbf{v}}}{\partial y}(\hat{\mathbf{v}} - \mathbf{V}) - \frac{1}{\rho}\Big(\frac{\partial \hat{p}}{\partial y}\Big)^* + \nu \mathrm{div}\frac{\partial \hat{\mathbf{v}}}{\partial y} + \mathbf{F}, \tag{4}$$

where $\mathbf{F}$ is the strength of the gravity field, ρ is the fluid density, $\nu = \mu/\rho$ is the kinematic viscosity coefficient [11].

And now, the boundary-value problem is reduced to finding the solution of a system of partial differential equations, namely, equation (4) plus the equation of continuity $\mathrm{div}\hat{\mathbf{v}} = 0$. This solution must satisfy the sticking condition $\hat{\mathbf{v}}(t,y)\Big|_S = \mathbf{V}$ and the natural condition $\lim\limits_{y\to\infty} \hat{\mathbf{v}}(t,y) = 0$.

A flow is accepted to call established or stationary if the field of its absolute velocity vectors in the moving coordinate system does not change in time. Obviously, if the body moves translationally, the necessary condition for the flow to be stationary is $\mathbf{V} = \mathbf{V}_0 = \mathrm{const}$.

The formulae for the power of the drag force acting upon a homogeneous solid sphere, in stationary cases considered by Stokes and Oseen, are presented below [12].

The Stokes procedure ignores in (4) the strength of the gravity field and the term $\dfrac{\partial \hat{\mathbf{v}}}{\partial y}(\hat{\mathbf{v}} - \mathbf{V})$. As a result, the expression for the drag force becomes $D = 6\pi\mu a V_0$, where V_0 is the magnitude of the velocity vector $\mathbf{V}_0$, and a is the radius of the solid sphere. For the purpose of forthcoming generalizations, it is convenient to rewrite this expression as follows:

$$D = C_D^{St}\rho S V_0^2/2, \qquad C_D^{St} = 24/\mathrm{Re}. \tag{5}$$

Here $S = \pi a^2$, C_D^{St} is the drag coefficient, and $\mathrm{Re} = 2aV_0/\nu$ is the Reynolds number.

Let $\mathbf{i}, \mathbf{j}$ be the unit vectors in the directions Ox and Oy respectively. We need further a mapping that puts a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ into correspondence to $\mathbf{a}^\perp = -a_2\mathbf{i} + a_1\mathbf{j}$. Let V be the magnitude of $\mathbf{V}$, D be that of the drag force, and D_l be that of the lift force. For needs of forthcoming references, it is convenient to formulate the following assertion as lemma.

Lemma 1. The drag and lift forces are calculated by the formulae

$$\mathbf{D} = \mathrm{sgn}(\mathbf{V},\mathbf{D}) D \frac{1}{V}\mathbf{V}, \quad \mathbf{D}_l = \mathrm{sgn}(\mathbf{V},\mathbf{D}) s D_l \frac{1}{V}\mathbf{V}^\perp, \tag{6}$$

$$s = \mathrm{sgn}((\mathbf{V},\mathbf{e})(\mathbf{V},\mathbf{e}^{\perp}))$$

where $\mathbf{e}$ is the directing vector of the body symmetry axis.

The magnitude of the drag force

$$D = C_D \rho S V^2/2. \tag{7}$$

Analogously, the magnitude of the stationary lift force can be presented as

$$D_l = C_D^{\perp} \rho S V^2/2. \tag{8}$$

Here S is the area of the body projection onto the plane perpendicular to the velocity vector of the body inertia center.

According to the theory of dynamic similitude, the coefficients C_D and $C_D^{\perp}$ depend on the body shape, Reynolds and Frud numbers only.

Let us introduce the following constraint.

Constraint 3. The body moves in a volume of fluid which is either very extended or is enclosed within rigid boundaries.

In the framework of the listed constraints, the coefficient C_D is a function of the body shape, Reynolds number and, probably, the angle of attack between the velocity vector of the body inertia center and the symmetry axis, i.e., $C_D = C_D(\text{shape,Re}, \alpha)$. To determine the angle of attack, one can use the formula

$$\alpha = -s \arccos |(\mathbf{e}, \mathbf{V}/V)|. \tag{9}$$

The work of the hydrodynamic forces is considered further as performance index. In [3] we show that solving the problem of optimal displacements of a solid sphere, if the flow is quasistationary, leads to the relative mistake in the optimal energy consumption about 3% only (Reynolds numbers are assumed to obey the restriction $\text{Re} < 1$). The nonstationarity of the flow can be partially taken into account by means of introducing the apparent additional mass.

Hypothesis 1. The optimal displacement of the body produces quasistationary flow.

APPLIED METHODS FOR SOLVING DYNAMIC FLOW OPTIMIZATION PROBLEMS

Let the control actions not explicitly enter into the expression for the power of resistance forces. Then the current value of the power of the resistance forces should be uniquely determined by the realized part of the phase trajectory of the system. In this situation, problems of dynamic optimization of the first type are reduced to classical auxiliary problems.

In such problems, dynamic constraints consist of an equation for the operation of the resistance forces and the kinematic relationships of a mechanical system. Derivatives of generalized coordinates take on the role of controls. Thus constructed auxiliary problem in form belongs to the number of problems of the classical calculus of variations.

For systems with a nonsmooth surface, for example, for cylindrical bodies, manifolds appear in the space of generalized coordinates and velocities of the original problem on which the projection of these bodies onto a plane perpendicular to the velocity vector of their center of mass, and therefore the Hamiltonian, loses its differentiability property. The optimal control forces and moments are found from the equations of dynamics of the systems under consideration.

To solve the problems of dynamic optimization of the second type, an engineering approach was chosen to overcome the above difficulties. The reduction carried out within the framework of this approach is based on the fact that the system moves in a potential field of gravity, and at the same time, part of the work of control forces and moments is spent on kinetic energy change.

Hence, the variable part of the work will coincide with the energy costs of overcoming the resistance forces in the following two cases. In the first of them, the boundary phase state is uniquely set for the system. In the second case, when the phase image of the system is required to reach a certain target set, the desired fact can be provided by limiting the allowable forces and moments, if only the system is completely controllable.

In this case, there are always suitable impulse control forces and moments, the use of which leads to the termination of the system at the last moment of the control process. These effects do not affect the total energy costs of overcoming the resistance forces. As a result, the problem of the second type is reduced to the problem of the first type. The described approach corresponds to a rigorous mathematical formalization of the above nonlinear operations on generalized functions.

Indeed, in order to justify this statement, it is necessary to solve the Lagrange equations of the system's motion relative to the control actions, then substitute the result in the expression for the power of the control forces and moments and use the definitions of multiplication of discontinuous functions by the impulse functions set forth in [3]. As a result, firstly, an idea will be obtained for the operation of control forces and moments in the form of a sum of kinetic and potential energy and work of resistance forces. Secondly, to describe the dynamics of the resistance forces, a first-order differential equation in the normal Cauchy form will be derived. It is clear that the right-hand side of this equation is uniquely determined by the phase state of the system. This allows us to choose only kinematic relationships as dynamic constraints. As a result, the initial problems will be reduced to the auxiliary ones indicated above. It should be emphasized that during the transition from initial problems to auxiliary ones, the constraints 1 and 2 turn into direct constraints on the controls. In conclusion, it should be noted that in the formulation of auxiliary problems, the constraints 1 and 2 are considered implicitly present. The reason for this is that the solutions to auxiliary problems, as a rule, coincide with the solutions to these problems without taking into account restrictions.

CONCLUSION

The features of dynamic dynamic energy optimization problems for a viscous medium flowing at a non-constant density of absolutely rigid bodies with variable geometry are described. Moreover, special attention is paid to the features of such problems from the point of view of the theory of optimal control, the difficulties caused by these features, and how to overcome them. The scheme for solving problems that fit into the general formulation, designed for the applicationist, is briefly described, and the idea of a rigorous mathematical justification of this scheme is presented.

ACKNOWLEDGMENTS

The investigation was supported by the Russian Foundation for Basic Research, project no. 19-01-00371-a.

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